Arid Zone Journal of Engineering, Technology and Environment, December, 2017; Vol. 13(6):790-800
Copyright © Faculty of Engineering, University of Maiduguri, Maiduguri, Nigeria.
Print ISSN: 1596-2490, Electronic ISSN: 2545-5818, www.azojete.com.ng

ALGORITHM FOR THE DYNAMIC ANALYSIS OF PLANE RECTANGULAR RIGID FRAME SUBJECTED TO GROUND MOTION

H. O. Ozioko and E. I. Ugwu

(Civil Engineering Department, Michael Okpara University of Agriculture Umudike, Abia State, Nigeria.)
Corresponding Author's email address: ho.ozioko@mouau.edu.ng

ABSTRACT
Dynamic analysis of frames involves very rigorous method which always requires software programming to take care of cumbersome calculations that are always involved. An easy manual method has been developed here to take care of these rigorous calculations that always require software programming. This method was used to analyze a three storey plane rectangular rigid frame. The condensed matrix gotten from this method was compared with that of Chopra and the result shows high similarity. Eigen value, eigenvector, shear force and base overturning moment of the three storey frame were also determined in the analysis. The results showed that plane rectangular rigid frames can easily be analyzed with this method which is less cumbersome and easy to apply. However this method cannot easily handle more than ten storey frames because of large matrix calculations involved.

Keywords: Dynamics, eigenvalues, overturning, eigenvectors and frames.

1.0 INTRODUCTION

Galileo was one of the first to deal thoroughly with the concept of acceleration and he founded dynamics as a branch of natural philosophy. The close interplay of theory and experiment, characteristic of this subject, was founded by Italian scientists. Galileo said mathematics is the means to decipher the book of nature. Mathematics seeks to discern the outlines of all possible abstract structure. This pure mathematics may be applied to every sort of concrete problem (Corradi, 2013).Structural dynamics deals with the structures that are subjected to dynamic loading. The loads include windstorm, earthquake loads, people, hurricanes, waves, vibration of the ground due to a blast nearby, flooding, tornadoes, vibration effect from compound disc player, vibrations effect from moving vehicles, vibration effect from heavy construction equipment etc. Any structure can be subjected to dynamic loading. Dynamic analysis can be used to find dynamic displacements, Eigen value, eigenvector, shear force and bending moment. (Ezeh *et al*, 2014). The problem of dynamic analysis is the rigorous and cumbersome calculations that one has to solve to obtain the Eigen values and eigenvectors. Recently, authors have developed software that can handle this cumbersome calculation but there is still need to develop manual method which could facilitate learning and manual analyses of these dynamic loads. This fall into the main objectives of this study which is the algorithm for the dynamic analysis of plane rectangular rigid frame subjected to ground motion. This objectives can be achieved through the following methods:

- Matrix stiffness method to analyse plane rectangular rigid frame.
- Method of static condensation which reduces the number of degrees of freedom of the plane rectangular rigid frame.
- Household characteristic equation, Newton-Raphson's method and Gaussian elimination method in the vibration analysis of plane rectangular rigid frame subjected to ground motion.
- Manually determination of the natural frequency of different mode shape (eigenvalues and eigenvectors of the frame subjected to ground motion).
- Solving for dynamic base shear force and dynamic overturning moment.

2.0 Materials and Methods

2.1 Elastic Frame Assumptions

i. Members of the frame are axially rigid (stiff). Hence, there are only two possible displacements per joint or node.
ii. The element of the frame are concieved as finite (lump) mass system.
iii. The nodal displacement of the frame are numbered as shown in Figure 1
iv. The U's stands for the lateral and rotational displacement
v. Axial deformation of beams, columns and the effect of axial forces on the stiffness of the columns were neglected. This is because the inertial effect associated with rotations and vertical displacements of the joints are usually not significant.(Clough and Penzien 1995)

2.1.1 Numbering of the Frame

The nodes, elements and the deformation or forces acting on the frame were numbered first from the base to the top, having in mind that a frame with N number of lateral deformation will also have 2N number of rotational deformation i.e figure 1 has 6 number of lateral deformations and 6 number of rotational deformations. This leaves every node with three DOFs which are:

i. Axial deformation
ii. Lateral deformation
iii. Rotational deformation.

With axial deformation neglected, the total deformations at each node reduces to two (lateral and rotational deformation).

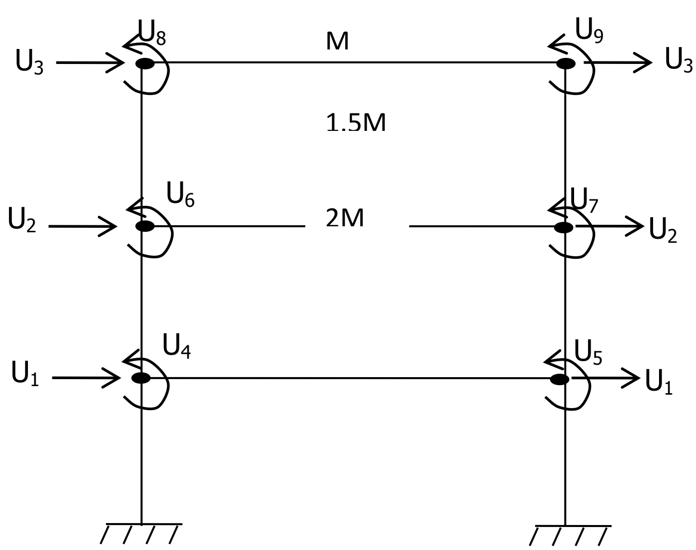


Figure 1: Three storey frame showing the mass, lateral and rotational displacement.

2.2 Method of static condensation elements and global stiffness matrix.

Roy and Chakrabarty, (2009) derived the element stiffness matrix of an axially rigid line elements with two nodes as:

$$\begin{bmatrix} P_i \\ M_i \\ P_j \\ M_j \end{bmatrix} = EI/L^3 \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{bmatrix} V_i \\ \theta_i \\ V_j \\ \theta_j \end{bmatrix} \quad (1)$$

Equation (1) was used to determine the elements stiffness matrices for every element of the frame. After which the various elements stiffness matrices were combined to form the global stiffness matrix.

Where P_i and P_j are the initial and final loads applied to the element and $M_i$and M_j is the initial and final moments generated from applied load on the element. V_i, V_j and θ_i, θ_j are the initial and final lateral and rotational deformations generated from the effect of the load respectively.

Static Condensation.

This method eliminate those degrees of freedom without masses assigned to them. This happens by eliminating the rotational deformations and leaving the structure with only lateral deformations. Thereby reducing the global matrix to one-third of its original size (Anderson and Naeim, 2012).Thus the global matrix was partitioned as shown in matrix A and the necessary condensation were carried out as shown.

$$\text{Matrix A} = \left[\begin{array}{c|c} K_{tt} & K_{t0} \\ \hline K_{0t} & K_{00} \end{array}\right]$$

k_{00}^{-1} and k_{0t}^{T} were determined from K_{00} and K_{0t} of matrix A and substituted in equation (2) to obtain the condensed matrix.

$$\hat{k}_{tt} = k_{tt} - k_{0t}^{T} k_{00}^{-1} k_{0t} \quad (2)$$

The steps are summarized below

i. Partition the global stiffness matrix like the static condesation pattern of matrix A.
ii. Obtain k_{0t}^{T} = k_{t0} and k_{00}^{-1}
iii. Obtain T = -$k_{00}^{-1} k_{0t}$ hence, -$k_{0t}^{T} k_{00}^{-1} k_{0t}$
iv. Substitute for $\hat{k}_{tt}$= k_{tt} - $k_{0t}^{T} k_{00}^{-1} k_{0t}$
v. $\hat{k}_{tt}$is the condensed stiffness matrix which ensures that the matrix has been reduced to one-third of its original size.

3.0 Vibration analysis using household characteristic equation and Newton-Raphsoniteretion method.

Eigenvalues are obtained by solving for w^2 in equation (3). Thus

$$\text{Det}(\hat{k}_{tt} - w^2 m_{tt}) \quad (3)$$

Where w^2 = ʎ = eigenvalue, m_{tt} = lump mass matrix.
Substituting the values of condensed matrix $\hat{k}_{tt}$and lump mass matrix m_{tt}in equation (3) and solving the determinant of equation (3) by applying Szilard 2004, household characteristic equation gives a polynimial equation of the structure. This polynomial equation is solved by applying Bird 2010, Newton – Raphsons iterations method for solving root of polynomial equation which will yield various eigenvalues of the frame.
Below are the illustration on how to generate the polynomial.

$$A = \det \begin{matrix} a_{11} - ʎ & a_{12} a_{13} & \ldots\ldots\ldots\ldots\ldots\ldots & a_{1n} \\ a_{21} a_{22} - ʎ & a_{23} & \ldots\ldots\ldots\ldots\ldots\ldots & a_{2n} \\ a_{31} a_{32} a_{33} - ʎ & & \ldots\ldots\ldots\ldots\ldots & a_{3n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n1} a_{n2} a_{n3} & & \ldots\ldots\ldots\ldots\ldots\ldots & a_{nn} - ʎ \end{matrix} \quad (4)$$

The household characteristic equation were derived by Szilard, (2004) as expressed.

$$ʎ^n + k_1 ʎ^{n-1} + k_2 ʎ^{n-2} + \ldots\ldots + k_{n-1} ʎ + k_n = 0 \quad (5)$$

Where coefficients k_1, k_2, …………k_n are defined by

$$k_1 = -T_1 \quad (6a)$$

$$k_2 = -\frac{1}{2}(k_1 T_1 + T_2) \quad (6b)$$

$$k_3 = -\frac{1}{3}(k_2 T_1 + k_1 T_2 + T_3) \quad (6c)$$

. .
. .
. .

$$k_n = -\frac{1}{n}(k_{n-1} T_1 + k_{n-2} T_2 + \ldots\ldots\ldots + k_1 T_{n-1} + T_n) \quad (6d)$$

In equation (6), T_1, T_2, T_3, …………$T_n$represent the traces of the matrices A^n, which are defined as the sum of the diagonal elements of the corresponding matrix. Thus we can write for matrix A

T_1 = trace A
T_2 = trace A^2
. . .
. . .
. . .
T_n = trace A^n

The trace of a matrix is the summation of the diagonal element in the matrix. Consider matrix A of equation (4).

$$T_1 = \text{trace } A = a_{11} + a_{22} + a_{33} + \ldots\ldots\ldots + a_{nn} \quad (7a)$$

$$T_2 = \text{trace } A^2 = a_{11}^2 + a_{22}^2 + a_{33}^2 + \ldots\ldots\ldots\ldots + a_{nn}^2 \quad (7b)$$

$$T_3 = \text{trace } A^3 = a_{11}^3 + a_{22}^3 + a_{33}^3 + \ldots\ldots\ldots\ldots + a_{nn}^3 \quad (7c)$$

. . . . . .
. . . . . .
. . . . . .

$$T^n = \text{trace } A^n = a_{11}^n + a_{22}^n + a_{33}^n + \ldots\ldots\ldots\ldots\ldots + a_{nn}^n \quad (7d)$$

Substituting the value of equation (7) into equation (6) yielded the value of k_1, k_2, k_3, ……..k_n. Thus the value of k_n and n for λ^n are substituted in equation (5) to obtain the polynomial equation of n degrees. The polynomial equation is solved by applying Newton – Raphson's iteration as explained.

3.1 Newton – Raphson's Iteration

In Bird (2010), the Newton – Raphson iteration popularly known as the Newton-Raphson's method, is stated as follows; "if r_1 is the approximate value of a real root of the equation f(x) = 0, then a closer approximation r_2, to the root is given by;

$$r_2 = r_1 - \frac{f(r_1)}{f^I(r_1)} \quad (8)$$

Equation (8) above is used to solve the polynomial equation of any degree by trial and error. Where

r_2 = root of the polynomial equation gotten by solving equation (8)
r_1 = assumed root for trials

$f(r_1)$ = the value obtained when r_1 is substituted in the polynomial equation.
$f^I(r_1)$ = the value obtained when r_1 is substituted in the differential of the polynomial equation.
A value for the root is assumed as r_1 and this value is substituted in the polynomial equation to obtain $f(r_1)$. The polynomial equation is differentiated and the same r_1 is substituted in the polynomial as the root to obtain $f^I(r_1)$. The result is then substituted in equation (8) to obtain r_2 and r_2 is subtstituted back in the polynomial to check if it is a true root of the polynomial equation. The true root of a polynomial equation will give zero when substituted in the equation. Continue with the same proceedure till all the roots of the polynomial are obtained.

3.2 Steps to follow when solving for the root of the polynomial equation using equation (8).

i. Choose a trial root value r_1, r_1 must be a positive integer since the eigenvalue can not be a negative integer.
ii. Obtain the differential of the polynomial equation of which its roots are required.
iii. Substitute the trial root value r_1 in the polynomial to obtain a value $f(r_1)$
iv. Substitute the trial root value r_1 in the differential of the polynomial to obtain another value $f^I(r_1)$
v. Substitute the value of r_1, $f(r_1)$ and $f^I(r_1)$ in equation (8) to obtain r_2.
vi. Check if r_2 will satisfy the polynomial equation by substituting r_2 in the polynomial equation to see if the polynomial will give zero.
vii. If r_2 did not satisfy the polynomial equation, then increase r_1 and repeat step 3 to 6.
viii. Continue step 7 until a value of r_2 that satisfy the polynomial is obtained and this value is one of the root of the polynomial depending on the degree of the polynomial.
ix. Increase r_1 again and repeat step iii to viii to obtain the second root of the polynomial equation.
x. Continue step 9 until all the roots of the polynomial are obatained (depending on the degree of the polynomial). These roots are the required eigenvalues.

3.3 How to solve for Eigenvector or Natural Mode Φ Using Gaussian Elimination Method.

The eigenvectors are obtained by solving equation (9) below.

$$(\hat{k}_{tt} - \Lambda_n m_{tt})\Phi_{nj} = 0 \qquad (9)$$

The number of eigenvalues in a system determine the number of eigenvectors as well. The number of eigenvectors in a system are the square of the number of eigenvalues provided the number is not one. Each eigenvalues generate eigenvector that equals the number of eigenvalues of the system. i.e. For the case of three story frame of figure (1), n = 3 and Λ_1 will generate Φ_{11}, Φ_{12}&Φ_{13}, Λ_2 will generate Φ_{21}, Φ_{22}&Φ_{23} and Λ_3 will generate Φ_{31}, Φ_{32}&Φ_{33}. From the last statement we noted that the system has three eigenvalues and nine eigenvectors.

Substituting the value of $\hat{k}_{tt,}\Lambda_n$&m_{tt} in equation (9) gives a matrix of n degrees. To obtain the eigenvectors, Gaussian elimination method is applied to the linear simultaneous equations i.e. one of the eigenvectors from the singular matrix has to be assumed to be one and then the fraction of others are solved for.

The set of the eigenvector simultaneus equation can be given as follows.

$$\begin{array}{l}
a_{11}\Phi_{11} + a_{12}\Phi_{12} + a_{13}\Phi_{13} + \ldots\ldots\ldots\ldots + a_{1n}\Phi_{1n} = 0 \\
a_{21}\Phi_{21} + a_{22}\Phi_{22} + a_{23}\Phi_{23} + \ldots\ldots\ldots\ldots + a_{2n}\Phi_{2n} = 0 \\
a_{31}\Phi_{31} + a_{32}\Phi_{32} + a_{33}\Phi_{33} + \ldots\ldots\ldots\ldots + a_{3n}\Phi_{3n} = 0 \\
\quad . \qquad . \qquad . \qquad\qquad\qquad . \quad . \\
\quad . \qquad . \qquad . \qquad\qquad\qquad . \quad . \\
\quad . \qquad . \qquad . \qquad\qquad\qquad . \quad . \\
a_{n1}\Phi_{n1} + a_{n2}\Phi_{n2} + a_{n3}\Phi_{n3} + \ldots\ldots\ldots\ldots + a_{nn}\Phi_{nn} = 0
\end{array}$$

$$\begin{bmatrix} a_{11}a_{12}a_{13} & \ldots\ldots\ldots\ldots & a_{1n} \\ a_{21}a_{22}a_{23} & \ldots\ldots\ldots\ldots & a_{2n} \\ a_{31}a_{32}a_{33} & \ldots\ldots\ldots\ldots & a_{3n} \\ . \quad . \quad . & & . \\ . \quad . \quad . & & . \\ . \quad . \quad . & & . \\ a_{n1}a_{n2}a_{n3} & \ldots\ldots\ldots\ldots & a_{nn} \end{bmatrix} \begin{bmatrix} \Phi_{n1} \\ \Phi_{n2} \\ \Phi_{n3} \\ . \\ . \\ . \\ \Phi_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} \qquad (10)$$

3.4 Steps to follow when solving for Eigenvectors

i. Let Φ_{nn} = 1
ii. Divide all the row by the coefficient of Φ_{n1}
iii. Subtract every other row from the first row to make the coefficient of Φ_{n1} of every other row apart from first row to be zero.
iv. Divide every other row apart from the first row by the coefficient of Φ_{n2} to make the coefficient of Φ_{n2} to be one (1).
v. Subtract every other rows apart from first row from the second row to make their coefficient Φ_{n2} to be zero.
vi. Divide every other row apart from first and second row by the coefficient of Φ_{n3} to make their coefficient Φ_{n3} to be one (1).
vii. Subtract every other rows apart from first and second row from the third row to make their coefficient Φ_{n3} to be zero.
viii. Continue in this manner till the last row become solvable.
ix. Repeat step 1 to 8 for all the eigenvalues (Λ_n, where n = 1, 2, 3, …)

Table 1: Modal Static and Dynamic Response, r

Modal Static Response, r	Modal Dynamic Response, r
V_i == $V_{in}^{st} = \sum_{j=1}^{N} s_{jn}$	V_i = $V_{in}^{st} A_n$(t)
M_i == $M_{in}^{st} = \sum_{j=1}^{N}(h_j - h_i)s_{jn}$	M_i = $M_{in}^{st} A_n$(t)
V_b == $V_{bn}^{st} = \sum_{j=1}^{N} s_{jn} = \Gamma_n L_n^h \equiv M_n^*$	V_b = $V_{bn}^{st} A_n$(t)
M_b == $M_{bn}^{st} = \sum_{j=1}^{N} h_j s_{jn} = \Gamma_n L_n^\theta \equiv h_n^* M_n^*$	M_b = $M_{bn}^{st} A_n$(t)
U_j == $u_{jn}^{st} = (\Gamma_n / w_n^2)\Phi_{jn}$	U_j = $u_{jn}^{st} A_n$(t)
Δ_j == $\Delta_{in}^{st} = \sum_{j=1}^{N} (\Gamma_n / w_n^2)(\Phi_{jn} - \Phi_{j-1,n})$	Δ_j = $\Delta_{in}^{st} A_n$(t)

Where $M_n^* = \Gamma_n L_n^h = \frac{(L_n^h)2}{M_n} h_n^* = \frac{(L_n^\theta)}{L_n^h} L_n^\theta = \sum_{j=1}^{N} h_j m_j \Phi_{jn}$

Where V_{bn} and M_{bn} are the base shear and overturning moment due to nth mode.

3.5 Overall steps to the present study

STEP1: Number the frame. When numbering the frame, number the joint separately, also number the two component members (beams and column) together and then number the displacements separately. For each joint, we numbered the lateral and rotational displacement.

STEP2: Determine the stiffness matrix. The stiffness matrix equation of equation (1) was used to determine the stiffness matrix of the frame. Each element has two nodes and every node has two deformations, lateral and rotational deformation. But in a beam element, the lateral deformations are neglected because it has been taken care of in the neighboring joint column.

STEP3: Add the stiffness matrices for all the elements to obtain the global stiffness matrix.

STEP4: Determine the condensed stiffness matrix. The condensed stiffness matrix is determined by the method of static condensation. This method eliminates from dynamic analysis those degrees of freedom of a structure to which zero mass is assigned.

STEP5: Determine the lump mass matrix.

STEP6: Determine the eigenvalue from the frequency equation (3). Solve equation (3) by applying Szilard household characteristic equation and Birds (2010) Newton-Ralphson procedures for solving root of polynomial equations.

STEP7: Determine the natural modes or Eigenvectors Φ_n by Gaussian elimination.

STEP8: Draw the graphical representation of the Eigenvectors.

STEP9: Determine the base shear forces.

STEP10: Determine the base overturning moments.
Chopra 2007, summarised the formular for calculating modal static and modal dynamic responses with tables as shown in Table 1.

4.0 Conclusions.

Based on the results of this research, the following conclusions were drawn;

i. A three storey Plane rectangular rigid frames subjected to ground motions was analyzed.
ii. The steps marshalled out in this work yielded good results when applied to dynamic frame.
iii. The method developed showed high level of accuracy when the result of the condensed matrix obtained with it was compared with that of Chopra.
iv. The method is an efficient and satisfactory method for dynamic analysis of frame subjected to ground motion.

5.0 Recommendations.

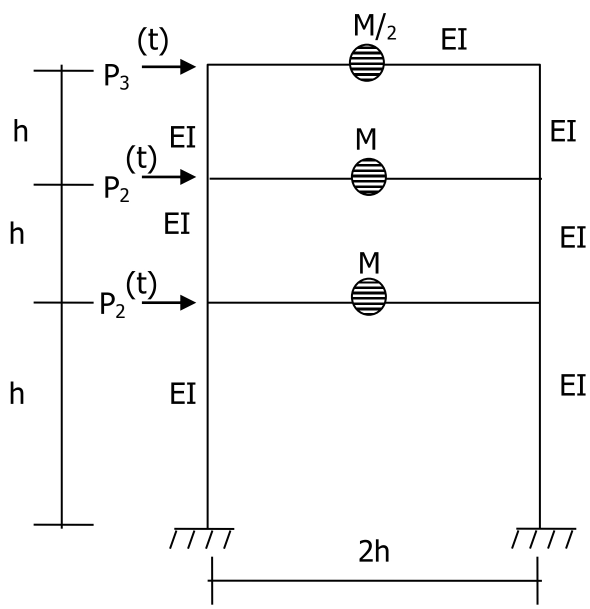
The following recommendations were made:

i. The develop highly recommends this method in the dynamic analysis of frames subjected to ground motion
ii. Future research work should consider using matrix stiffness method to solve for rectangular frames that has consistence mass and compare its convergence with plane rectangular frames that has lump mass.

APPENDICE 1

Example:

i. The figures "a" below showed a three storey frame with lumped masses subjected to lateral forces, together with its properties; in addition, the flexural rigidities are given in EI's for columns and beams. Determine the condensed matrix. Neglect the axial deformation of the members. Assume the members to be massless and determine the eigenvectors and overturning moments.



Appendix 1: Three storey rectangular plane rigid frame

Solution

Equation (1) was used to determine the elements stiffness matrices for every element of the frame. After which the various elements stiffness matrices were combined to form the global stiffness matrix thereby forming a nine by nine global matrix as shown.

Matrix Table

1	2	3	4	5	6	7	8	9	
48	-24	0	0	0	-6	-6	0	0	1
-24	48	-24	6	6	0	0	-6	-6	2
0	-24	24	0	0	6	6	6	6	3
0	6	0	10	1	2	0	0	0	4
0	6	0	1	10	0	2	0	0	5
-6	0	6	2	0	10	1	2	0	6
-6	0	6	0	2	1	10	0	2	7
0	-6	6	0	0	2	0	6	1	8
0	-6	6	0	0	0	2	1	6	9

Condensed Matrix

The global matrix was partitioned as shown in matrix A and the necessary condensation were carried out using equations (2). The results were shown below.

40.84645161	-23.25677	5.109677
-23.25677419	31.091613	-14.2452
5.109677419	-14.24516	10.06452

Solving For Eigenvalues.

The mass matrix and condensed matrix were substituted in equation (3) to obtain the eigenvalues as shown below

Mass Matrix.

1	0	0
0	1	0
0	0	0.5

Eigenvalue 1

38.72185161	-23.25677	5.109677
-23.25677419	28.967013	-14.2452
5.109677419	-14.24516	9.002216

Eigenvalue 2

28.64015161	-23.25677	5.109677
-23.25677419	18.885313	-14.2452
5.109677419	-14.24516	3.961366

Eigenvalue 3

18.11095161	-23.25677	5.109677
-23.25677419	8.3561129	-14.2452
5.109677419	-14.24516	-1.30323

Solving for Eigenvectors

The eigenvectors shown are obtained by solving equation (9) .

Eigenvector Graph

Graph (Y)	Mode 1 (X)	Mode 2 (X)	Mode 3 (X)
0	0	0	0
4	0.2708	-0.0142	-1.0115
7	0.6589	0.2028	-0.5245
10	1	1	1

Solving for base shear force and overturning moment

The modal properties, base shear force due to nth mode and overturning moment due to nth mode are obtained with the help of table (1).

The Modal Properties

S/N	Mn	Ln	Rn	Vb
1	1.01	1.43	1.419	2.029
2	0.54	0.689	1.272	0.876
3	1.8	-1.04	-0.576	0.597

Base shear due to nth mode

S/N	Vbbn
1	2.03
2	0.88
3	0.6

Overturning moment due to nth mode

S/N	L	Mbn
1	6.45	9.15
2	3.77	4.794
3	-2.13	1.229

Efficacy of the method for condensed matrices

Chopra (2007), showed the condensed matrix of a three storey plane rectangular frame with lumped masses subjected to lateral forces, together with its properties; in addition, the flexural rigidity is EI for columns and beams. The problem was given in appendix 1 (Example).

Chopra condensed matrix (matrix C) are listed below

$$C = \begin{bmatrix} 40.85 & -23.26 & 5.11 \\ -23.26 & 31.09 & -14.25 \\ 5.11 & -14.25 & 10.06 \end{bmatrix}$$

Present Method condensed matrix (matrix D) are listed below

$$D = \begin{bmatrix} 40.8465 & -23.2568 & 5.10968 \\ -23.25677 & 31.0961 & -14.2452 \\ 5.109677 & -14.2452 & 10.0645 \end{bmatrix}$$

From the result of the condensed matrix obtained, it was observed that Chopra's solution and the present study values for the condensed matrix yielded approximately the same values.
Thus, the method is satisfactory and can be relied upon in dynamic analysis.

References

Anderson, JC. and Naeim, F. 2012. Basic Structural Dynamics. MC Graw – Hill, New York.

Bird, J. 2010. Higher Engineering Mathematics, 6th Ed. Elsevier Ltd, New York.

Chopra, AK. 2007. Dynamics of Structures: Theory and Applications to Earthquake Engineering. Second edition, Prentice hall Eaglewood Cliffs, New Jersey.

Clough, RW. and Penzien, J. 1995. Dynamics of Structures.Third edition MC Graw – Hill, New York.

Corradi, M. 2013. A Short Account of the History of Structural Dynamics Between the Nineteenth and Twentieth Centuries. www.arct.cam.ac.uk/Downloads/ichs/vol-1-corradi.pdf. Internet

Ezeh JC., Ibearugbulem O., Ezeokpube GC. and Ozioko HO. 2014. Dynamics analysis of two storey plane rectangular rigid frame subjected to ground motion. International Journal of Engineering Science and Research Technology, November, Volume 1, No. 2

Glyn, J. 2011. Advanced Engineering Mathematics, 4th Edition Pearson Educational Ltd.

Roy, SK. and Chakrabarty, S. 2009. Fundamental of Structural Analysis with Computer Analysis and Applications. S. Chand and Company Limited, Ram Nagar, New Delhi.

Stroud, KA. and Dexter, JB. 2011. Advanced Engineering Mathematics. 5th Ed. Palgrave Macmillan, New York.

Szilard, R. 2004. Theories and Applications of Plate Analysis. John Wiley and Sons Inc, New Jersey.